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Approximation by Certain Subspaces in the Banach Space of Continuous Vector-Valued Functions*

DAN AMIR

Department of Mathematical Sciences, Tel-Aviv University, Tel-Aviv, Israel

AND

FRANK DEUTSCH

Department of Mathematics, Pennsylvania State University, University Park, Pennsylvania 16802

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A theory of best approximation is developed in the normed linear space $C(T, E)$, the space of E-valued bounded continuous functions on the locally compact Hausdorff space T, with the supremum norm. The approximating functions belong to the subspace $C_F(T, E)$ of $C(T, E)$ consisting of those functions which have "limit at infinity" which lies in the subspace F of the normed linear space E. A distance formula is obtained, and a selection for the metric projection onto $C_F(T, E)$ is constructed which has many desirable properties. The theory includes study of best approximation in l_∞ by the subspace c_0, and closely parallels the known theory of best approximation by M-ideals (although our subspace is not an M-ideal, in general).

1. INTRODUCTION

The starting point for this paper was our discovery that the problem of best approximation in ℓ_∞ by the subspace c_0 has a very rich, detailed, and complete theory associated with it. Examples are, the simple distance formula for an element $x \in \ell_\infty : d(x) = d(x, c_0) = \lim_n \sup |x(n)|$; and the function $\sigma : \ell_\infty \to c_0$, defined by $(\sigma x)(n) = 0$ if $|x(n)| \leqslant d(x)$ and $(\sigma x)(n) = [1 - d(x)/|x(n)|]\, x(n)$ otherwise, which is a homogeneous, Lipschitz continuous, selection for the metric projection P_{c_0}, and which has the minimal norm property: $\| \sigma x \| = \min\{\| y \| : y \in P_{c_0}x\}$. Given any $x \in \ell_\infty \backslash c_0$, c_0 is the cone generated by $P_{c_0}x - P_{c_0}x$ (showing that $P_{c_0}x$ is rather "fat").

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We next observed that our results (and proofs) carried over to the more general situation of best approximation in $C(T, E)$—the space of bounded continuous E (a normed linear space)-valued functions x on a locally compact Hausdorff space T with the norm $\| x \| = \sup\{\| x(t)\| : t \in T\}$—by the subspace $C_0(T, E)$ of those functions "vanishing at infinity" (precise definitions will be given below). Since $C_0(T, E)$ is an "M-ideal" in $C(T, E)$ (Proposition 4.4). there is a substantial theory that is already known (cf. [7] and [5]). However, even where there is some overlap with the known results, we have obtained, in general, stronger, more detailed results, whose proofs are more elementary.

Finally, we were able to extend all our results to the still more general setting of best approximation in $C(T, E)$ by the subspace $C_F(T, E)$ of those functions which have "limit at infinity" in the subspace F of E. (Here we assume that E is "uniformly convex with respect to F.") Moreover, since $C_F(T, E)$ is *not* an M-ideal in $C(T, E)$ in general (Proposition 4.5), the M-ideal theory is of no help here. What is perhaps surprising then is that so much of the theory, valid for M-ideals, carries over to this situation.

For the remainder of the Introduction, we give the main definitions and notation to be used, and summarize the results to be proved.

Let T be a locally compact noncompact Hausdorff space and $\mathscr{K}$ the family of its compact subsets, directed by inclusion. Let E be a normed linear space and F a complete (linear) subspace of E. Consider the space $X = C(T, E)$ of bounded continuous E-valued functions x on T, with the norm $\| x \| = \sup \{\| x(t)\| : t \in T\}$, and its closed linear subspace $M = C_F(T, E)$ of functions x in X such that $x(\infty) \equiv \lim_{t\to\infty} x(t)$ exists and belongs to F. ($\lim_{t\to\infty} x(t) = e$ means that $\{t \in T : \| x(t) - e \| \geq \epsilon\} \in \mathscr{K}$ for every $\epsilon > 0$.) For any x in X, let $d(x) = d(x, M) = \inf\{\| x - y \| : y \in M\}$ denote the distance from x to M, and $Px = P_M x = \{y \in M : \| x - y \| = d(x)\}$ the (possibly empty) set of best approximations in M to x. The set-valued mapping $P = P_M : X \to 2^M$ is called the *metric projection* onto M.

The computation of $d(x)$, as well as the construction of a selection σ for P, involve the notions of relative Chebyshev radius and relative Chebyshev centers. If A is a bounded set in a normed linear space Y, we denote $r(y, A) = \sup\{\| y - a \| : a \in A\}$, $y \in Y$. For any subset G of Y, we define the *relative Chebyshev radius* of A with respect to G to be $r_G(A) = \inf\{r(y, A) : y \in G\}$, and the set of *Chebyshev centers* for A in G to be $Z_G(A) = \{y \in G : r(y, A) = r_G(A)\}$. (When A is a single point x, these notions reduce to the distance from x to G and the set of best approximations of x in G, i.e. $r_G(x) = d(x, G)$ and $Z_G(x) = P_G x$.)

If F is a subspace of the normed space E, then it is easy to verify that $Z_F(A)$ is a closed convex subset of F, $Z_F(\overline{co}(A)) = Z_F(A)$, and $Z_F(\alpha A) = \alpha Z_F(A)$ for every scalar α (where $\overline{co}(A)$ denotes the closed convex hull of A).

If F is a subspace of the normed space E, we say that E is *uniformly convex with respect to F* iff whenever x_n, y_n are such that $x_n - y_n \in F$, $\| x_n \| =$

$\| y_n \| = 1$ and $\| \frac{1}{2}(x_n + y_n)\| \to 1$, it follows that $x_n - y_n \to 0$. This is equivalent to the relative modulus of convexity

$$\delta_F(\epsilon) = \inf\{1 - \| \tfrac{1}{2}(x + y)\| : \| x \| = \| y \| = 1, \| x - y \| \geqslant \epsilon, x - y \in F\}$$

being positive for every $\epsilon > 0$. In particular, this implies that F must itself be uniformly convex (in the ordinary sense). Thus if F is complete, it is reflexive, hence boundedly weakly compact (i.e. F intersects each closed ball in E in a weakly compact set). There are two trivial examples of spaces such that E is uniformly convex with respect to F:

(1) E a uniformly convex Banach space, and F any closed subspace;

(2) E any normed linear space and $F = \{0\}$, the trivial subspace. In this case, $C_F(T, E)$ is the well known space $C_0(T, E)$ consisting of those $x \in C(T, E)$ vanishing at infinity.

In addition, there are examples which do not fall into either of these two classes, e.g.

(3) Let E be a normed space which is "uniformly convex in every direction" (u.c.e.d.), i.e. uniformly convex with respect to every one dimensional subspace, and let F be any finite dimensional subspace.

The fact that E is uniformly convex with respect to F in this case follows from a simple compactnesss argument and the following.

1.1. LEMMA (Day, James, Swaminathan). *E is uniformly convex in the direction of z iff* $\| x_n \| \leqslant 1$, $\| y_n \| \leqslant 1$, $x_n - y_n \to \lambda z$ *and* $\| \frac{1}{2}(x_n + y_n)\| \to 1$ *implies* $\lambda z = 0$.

The proof of the nontrivial implication is rather tedious and can be found in [3].

It is known [9] that every separable normed space has an equivalent u.c.e.d. norm, while only certain reflexive (viz. superreflexive) Banach spaces have an equivalent uniformly convex norm.

In Section 2 we give some properties of relative Chebyshev centers in relatively uniformly convex spaces. Section 3 contains the main results of the paper. Here we construct a continuous selection for the metric projection which has many "nice" properties (Propositions 3.4 and 3.6). Indeed, in a certain sense, a "nicer" selection is probably not available. In Section 4 we specialize to the important case when E is any normed space and $F = \{0\}$. (I.e. $X = C(T, E)$ and $M = C_0(T, E)$. This includes of course the approximation of ℓ_∞ by c_0.) In this case the results become much simpler and stronger. $C_0(T, E)$ is an "M-ideal" in $C(T, E)$ (in the sense of Alfsen and Effros [1]); for this particular M-ideal, our results are improvements upon results of Fakhoury [5] and Holmes, Scranton, and Ward [7] established for arbitrary

M-ideals. We also obtain an answer (Proposition 4.6) to one of the questions posed in [7], and a partial answer to another (cf. the paragraph preceding Proposition 4.4).

2. Some Properties of Relative Chebyshev Centers in Relatively Uniformly Convex Spaces

The following results, summarized in Lemma 2.1, are obtained by repeating almost verbatim known results in the non-relative case (i.e. when $F = E$). We shall produce them here for the sake of completeness.

2.1. Lemma. *Let F be a complete subspace of the normed space E, and E be uniformly convex with respect to F. Then every nonempty bounded subset A of E has a unique relative Chebyshev center $Z_F(A)$ in F, and the mapping $A \to Z_F(A)$ is uniformly continuous on $\{A : r_F(A) \leqslant R\}$ (for every R) in its Hausdorff semi-metric* $d_H(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\}$.

Proof. The existence of relative Chebyshev centers for A in F follows from the bounded weak compactness of F: if $y_n \in F$ are such that $r_F(y_n, A) \to r_F(A)$, then the (y_n) are bounded and we can take a w-convergent subsequence $y_{n_k} \to^w y$, and then for every $a \in A$ we have $\| a - y \| \leqslant \underline{\lim} \| a - y_n \| \leqslant \underline{\lim}\, r_F(y_n, A) = r_F(A)$, i.e. $r_F(y, A) \leqslant r_F(A)$ and necessarily $r_F(y, A) = r_F(A)$. (The same argument works for any reflexive subspace F or for every w^*-closed subspace F of a dual space E).

The uniqueness of relative Chebyshev centers follows from (and in fact, is equivalent to) the weaker assumption that E is uniformly convex with respect to every one-dimensional subspace of F: if z_1, z_2 are both in $Z_F(A)$, so is $z_0 = \frac{1}{2}(z_1 + z_2)$. Choose $x_n \in A$ with $\| x_n - z_0 \| \to r(z_0, A) = r_F(A)$. Then, necessarily, $\| x_n - z_i \| \to r_F(A)$ for $i = 1, 2$. Let $u_i^n = [1/r_F(A)](x_n - z_i)$. Then $\| u_i^n \| \leqslant 1$, $u_1^n - u_2^n = [1/r_F(A)](z_2 - z_1)$ and $\| u_1^n + u_2^n \| = [2/r_F(A)]\| x_n - z_0 \| \to 2$ so that E is not uniformly convex in the z-direction.

Finally we show the local uniform continuity of $A \to Z_F(A)$: Given $R > 0$ and $\epsilon > 0$, we may assume $\epsilon < 1$ and take $\eta > 0$ so that $\eta < (\epsilon/8)\, \delta_F(\epsilon/(R + 2))$. Let $r_F(A) \leqslant R$, $r_F(B) \leqslant R$, $d_H(A, B) < \eta$ and let $z = Z_F(A)$, $w = Z_F(B)$. We will show that $\| z - w \| < \epsilon$. For each $a \in A$, choose $b \in B$ so that $\| a - b \| < \eta$. Hence $\| a - w \| \leqslant \| a - b \| + \| b - w \| < \eta + r_F(B)$ implies $r_F(A) \leqslant \eta + r_F(B)$. By symmetry, $r_F(B) \leqslant \eta + r_F(A)$. Thus $\| a - w \| \leqslant 2\eta + r_F(A)$ and $\| z - w \| \leqslant 2\eta + 2r_F(A)$. If $r_F(A) \leqslant \epsilon/4$, $\| z - w \| < \epsilon$. If $r_F(A) > \epsilon/4$, consider $x = (a - z)/[2\eta + r_F(A)]$ and $y = (a - w)/[2\eta + r_F(A)]$. Then $\| x \| \leqslant 1$, $\| y \| \leqslant 1$, $x - y \in F$, and if $\| z - w \| \geqslant \epsilon$, then $\| x - y \| \geqslant \epsilon/(R + 2)$ implies

$$\begin{aligned}\| a - \tfrac{1}{2}(z + w)\| &= [r_F(A) + 2\eta] \| \tfrac{1}{2}(x + y)\| \\ &\leqslant [r_F(A) + 2\eta]\left[1 - \delta_F\left(\frac{\epsilon}{R+2}\right)\right] \\ &< r_F(A)\left[1 + \delta_F\left(\frac{\epsilon}{R+2}\right)\right]\left[1 - \delta_F\left(\frac{\epsilon}{R+2}\right)\right] \\ &< r_F(A)\end{aligned}$$

which contradicts $\sup_{a\in A} \| a - \frac{1}{2}(z + w)\| \geqslant r_F(A)$. Thus $\| z - w \| < \epsilon$. ∎

(In fact, the uniform convexity of E with respect to F is also necessary in order that $A \to Z_F(A)$ be uniformly continuous on subsets of the unit ball of E, cf. [2]).

2.2. Lemma. *If F is a complete subspace of the normed space E and E is uniformly convex with respect to F, then for every nonincreasing net (A_α) of nonempty bounded sets in E with $r_F(A_\alpha) \leqslant R$ for all α, the net $z_\alpha = Z_F(A_\alpha)$ converges.*

Proof. We may assume $R \geqslant 1$. $r_F(A_\alpha)$ is nonincreasing and converges to some $r \geqslant 0$. Let $\epsilon \in (0, 1)$ be given. If $r = 0$, take α with $r_F(A_\beta) < \frac{1}{2}\epsilon$ for $\beta > \alpha$ and then for $\gamma > \beta > \alpha$ and any $a \in A_\gamma$ we have $\| z_\beta - z_\gamma \| \leqslant \| z_\beta - a \| + \| a - z_\gamma \| \leqslant r_F(A_\beta) + r_F(A_\gamma) < \epsilon$, so that (z_α) is a Cauchy net.

If $r > 0$, take α with $r_F(A_\alpha) < r/[1 - \delta_F(\epsilon/R)]$. Then for every $\gamma > \beta > \alpha$ and every $a \in A_\gamma$ we have $\| a - z_\gamma \| \leqslant r_F(A_\gamma) \leqslant r_F(A_\alpha)$, $\| a - z_\beta \| \leqslant r_F(A_\beta) \leqslant r_F(A_\alpha)$, so that $\| z_\beta - z_\gamma \| \geqslant \epsilon$ implies $\| a - \frac{1}{2}(z_\beta + z_\gamma)\| = \frac{1}{2} \|(a - z_\beta) + (a - z_\gamma)\| \leqslant r_F(A_\alpha)(1 - \delta_F(\epsilon/r_F(A_\alpha))) \leqslant r_F(A_\alpha)(1 - \delta(\epsilon/R)) < r \leqslant r_F(A_\gamma)$, which shows $r(\frac{1}{2}(z_\beta + z_\gamma), A_\gamma) < r_F(A_\gamma)$, a contradiction. Therefore $\| z_\beta - z_\gamma \| < \epsilon$ for all $\gamma > \beta > \alpha$, and (z_α) is a Cauchy net. Since F is complete, (z_α) converges. ∎

3. The Selection σ

Throughout this section, unless explicitly stated otherwise, T will denote a locally compact noncompact Hausdorff space, E a normed linear space which is uniformly convex with respect to a complete subspace F, $X = C(T, E)$, $M = C_F(T, E)$, $P = P_M$, and $d(x) = d(x, M)$.

We proceed first to compute the distance $d(x)$, then define the selection σx for Px and study its properties.

3.1. Proposition. *For any subspace F of the normed space E and each $x \in X$,*

$$d(x) = \inf\{r_F(x(T\backslash K)) : K \in \mathscr{K}\}.$$

Proof. Denote $r(x) = \inf\{r_F(x(T\backslash K)) : K \in \mathscr{K}\}$. Given any $y \in M$ and $\epsilon > 0$, take $K \in \mathscr{K}$ with $\|y(t) - y(\infty)\| < \epsilon$ for all $t \notin K$. Then

$$\begin{aligned}\|x - y\| &\geqslant \sup\{\|x(t) - y(t)\| : t \notin K\}\\ &\geqslant \sup\{\|x(t) - y(\infty)\| - \epsilon : t \notin K\}\\ &= r(y(\infty), x(T\backslash K)) - \epsilon \geqslant r_F(x(T\backslash K)) - \epsilon\\ &\geqslant r(x) - \epsilon.\end{aligned}$$

Since ϵ was arbitrary, $\|x - y\| \geqslant r(x)$ for all $y \in M$ so $d(x) \geqslant r(x)$.

Conversely, given $\epsilon > 0$, take $K_0 \in \mathscr{K}$ with $r_F(x(T\backslash K_0)) < r(x) + \epsilon$. Then take $y_0 \in F$ with $r(y_0, x(T\backslash K_0)) < r(x) + \epsilon$. Let K_1 be a compact neighborhood of K_0 and let f be a continuous function on T satisfying $f(K_0) = 0 \leqslant f \leqslant 1 = f(T\backslash K_1)$. Let $y(t) = x(t) + f(t)[y_0 - x(t)]$. Then $y \in M$ (since $y(t) = y_0$ off K_1) and

$$\begin{aligned}\|x - y\| &= \sup\{f(t)\|y_0 - x(t)\| : t \in T\}\\ &= \sup\{f(t)\|y_0 - x(t)\| : t \notin K_0\}\\ &\leqslant \sup\{\|y_0 - x(t)\| : t \notin K_0\} < r(x) + \epsilon.\end{aligned}$$

Since ϵ was arbitrary, $d(x) \leqslant r(x)$. ∎

The *kernel* of the metric projection P_M is defined by

$$P_M^{-1}(0) = \{x \in X: 0 \in P_M x\} = \{x \in X: \|x\| = d(x, M)\}.$$

It is easy to see that $P_M^{-1}(0)$ is a closed and proper "cone", i.e. $\lambda x \in P_M^{-1}(0)$ whenever $x \in P_M^{-1}(0)$ and $\lambda \geqslant 0$. It is usually the case that the kernel of the metric projection onto a proximinal, but not Chebyshev, subspace has an interior. In spite of this, we have

3.2. PROPOSITION. *If F is any subspace of the normed space E, then $P_M^{-1}(0)$ is nowhere dense.*

Proof. It suffices to show that $P_M^{-1}(0)$ contains no ball centered at some $x \in P_M^{-1}(0)\backslash\{0\}$. Let $0 < \epsilon < \|x\|$ and choose $t_0 \in T$ such that $\|x(t_0)\| > \|x\| - \epsilon/2$. Choose a compact neighborhood K of t_0 and a continuous function f on T satisfying $f(T\backslash K) = 0 \leqslant f \leqslant 1 = f(t_0)$. Set $z = x + \epsilon x(t_0)f/[2\|x(t_0)\|]$. Then $z \in X$, $z - x \in C_0(T, E) \subset M$, and $\|z - x\| \leqslant \epsilon/2 < \epsilon$. Further,

$$\|z\| \geqslant \|z(t_0)\| = \|x(t_0)\| + \epsilon/2 > \|x\| = d(x) = d(z)$$

so $z \notin P_M^{-1}(0)$. ∎

3.3. DEFINITION. For each $x \in X$, we define

$$z(x) = \lim_{K \in \mathscr{K}} Z_F(x(T \setminus K)).$$

By Lemma 2.1, the mapping $x \in X \to z(x) \in F$ is well-defined, homogeneous, and unformly continuous on bounded sets.

For each $x \in X$ we also define

$$\begin{aligned}(\sigma x)(t) &= z(x) \qquad \text{if} \quad \| x(t) - z(x)\| \leqslant d(x) \\ &= z(x) + \left[1 - \frac{d(x)}{\| x(t) - z(x)\|}\right] [x(t) - z(x)] \qquad \text{otherwise.}\end{aligned}$$

3.4. PROPOSITION. *The mapping $\sigma : X \to M$ is a selection for the metric projection P_M which is idempotent, homogeneous, uniformly continuous on bounded sets, and (if $F \neq E$) nonlinear. More precisely*:

(1) $\sigma x \in P_M x$ *for each* $x \in X$;

(2) $\sigma^2 = \sigma$;

(3) $\sigma(\alpha x) = \alpha\sigma(x)$ *for each scalar* α;

(4) $\| \sigma x - \sigma y \| \leqslant 2 \| x - y \| + 2 \| z(x) - z(y)\|$;

(5) $\| \sigma x - z(x)\| \leqslant \| x - z(x)\| - d(x)$;

(6) $\| \sigma x - z(x)\| = \| x - z(x)\| \Leftrightarrow x \in M$;

(7) *There are* $x \in X$ *and* $y \in M$ *such that* $\sigma(x + y) \neq \sigma x + \sigma y$.

Proof. (1) Clearly, $\sigma x \in C(T, E)$. Given $\epsilon > 0$ choose $K \in \mathscr{K}$ such that $r_F(x(T\setminus K)) < d(x) + \frac{1}{2}\epsilon$ and $\| z(x) - Z_F(x(T\setminus K)) \| < \frac{1}{2}\epsilon$. Then for $t \notin K$ we have

$$\begin{aligned}\| x(t) - z(x)\| &< \| x(t) - Z_F(x(T\setminus K))\| + \tfrac{1}{2}\epsilon \\ &\leqslant r_F(x(T\setminus K)) + \tfrac{1}{2}\epsilon < d(x) + \epsilon;\end{aligned}$$

hence $\| \sigma x(t) - z(x)\| < \epsilon$ (for if $\| x(t) - z(x)\| \leqslant d(x)$, $\sigma x(t) = z(x)$, otherwise $\| \sigma x(t) - z(x)\| = \| x(t) - z(x)\| - d(x) < \epsilon$). Thus $\sigma x \in M$. Moreover, $\| x(t) - \sigma x(t)\| \leqslant d(x)$ for all t (since if $\| x(t) - z(x)\| \leqslant d(x)$, then $\sigma x(t) = z(x)$ by definition; while if $\| x(t) - z(x)\| > d(x)$, then $x(t) - \sigma x(t) = d(x)(x(t) - z(x))/\| x(t) - z(x)\|$). Thus $\sigma x \in Px$.

(2) If $x \in M$, then by (1) $\sigma x \in Px = x$, i.e. $\sigma x = x$. In particular, since $\sigma x \in M$ by (1), $\sigma^2 x = \sigma x$.

(3) This is immediate from the homogeneity of $z(x)$ and the absolute homogeneity of $d(x)$.

(4) We have to distinguish between 3 cases.

Case 1. $\|x(t) - z(x)\| \leqslant d(x)$ and $\|y(t) - z(y)\| \leqslant d(y)$. In this case we have $\sigma x - \sigma y = z(x) - z(y)$.

Case 2. $\|x(t) - z(x)\| > d(x)$ and $\|y(t) - z(y)\| > d(y)$. By symmetry we may assume that $d(x)/\|x(t) - z(x)\| \geqslant d(y)/\|y(t) - z(y)\|$. Then

$$
\begin{aligned}
\|\sigma x(t) - \sigma y(t)\| \leqslant{}& \|z(x) - z(y)\| \\
&+ \left\|\left[1 - \frac{d(x)}{\|x(t) - z(x)\|}\right][x(t) - z(x) - y(t) + z(y)]\right\| \\
&+ \left[\frac{d(x)}{\|x(t) - z(x)\|} - \frac{d(y)}{\|y(t) - z(y)\|}\right]\|y(t) - z(y)\| \\
\leqslant{}& \|z(x) - z(y)\| \\
&+ \left[1 - \frac{d(x)}{\|x(t) - z(x)\|}\right][\|x(t) - y(t)\| + \|z(x) - z(y)\|] \\
&+ d(x)\frac{\|y(t) - z(y)\|}{\|x(t) - z(x)\|} - d(y) \\
\leqslant{}& \|z(x) - z(y)\| + \left[1 - \frac{d(x)}{\|x(t) - z(x)\|}\right]\|z(x) - z(y)\| \\
&+ \left[1 - \frac{d(x)}{\|x(t) - z(x)\|}\right]\|x(t) - y(t)\| \\
&+ \frac{d(x)}{\|x(t) - z(x)\|}[\|y(t) - x(t)\| \\
&+ \|x(t) - z(x)\| + \|z(x) - z(y)\|] - d(y) \\
={}& 2\|z(x) - z(y)\| + \|x(t) - y(t)\| + d(x) - d(y) \\
\leqslant{}& 2\|z(x) - z(y)\| + 2\|x - y\|.
\end{aligned}
$$

Case 3. We may assume without loss of generality, that $\|x(t) - z(x)\| > d(x)$ and $\|y(t) - z(y)\| \leqslant d(y)$. Then

$$
\begin{aligned}
\|\sigma x(t) - \sigma y(t)\| ={}& \left\|z(x) + \left[1 - \frac{d(x)}{\|x(t) - z(x)\|}\right][x(t) - z(x)] - z(y)\right\| \\
\leqslant{}& \|z(x) - z(y)\| + \left[1 - \frac{d(x)}{\|x(t) - z(x)\|}\right]\|x(t) - z(x)\| \\
\leqslant{}& \|z(x) - z(y)\| + \|x(t) - z(x)\| - d(x) \\
\leqslant{}& \|z(x) - z(y)\| + \|x(t) - y(t)\| \\
&+ \|y(t) - z(y)\| + \|z(y) - z(x)\| - d(x) \\
\leqslant{}& 2\|z(x) - z(y)\| + \|x(t) - y(t)\| + d(y) - d(x) \\
\leqslant{}& 2\|z(x) - z(y)\| + 2\|x - y\|.
\end{aligned}
$$

This concludes the proof of (4).

(5) Since $z(x) \in F$, it is also in M (regarded as a constant function on $T: z(x)(t) = z(x)$). Hence $\| x - z(x)\| \geqslant d(x)$. If $\sigma x(t) \neq z(x)$, then

$$\begin{aligned}\| \sigma x(t) - z(x)\| &= \| x(t) - z(x)\| - d(x) \\ &\leqslant \| x - z(x)\| - d(x).\end{aligned}$$

Thus this inequality holds for all t and (5) is proved.

(6) If $x \in M$, $\sigma x = x$ and hence $\| \sigma x - z(x)\| = \| x - z(x)\|$. Conversely, if $x \notin M$, then $d(x) > 0$ and (5) implies $\| \sigma x - z(x)\| < \| x - z(x)\|$.

(7) Choose any vector $e \in E \backslash F$ such that $\| e \| = 1 = d(e, F)$ and define $x(t) = e$ for all $t \in T$. Then

$$\begin{aligned}1 = \| x \| \geqslant d(x) &= \inf_{y \in M} \| x - y \| = \inf_{y \in M} \sup_{t \in T} \| e - y(t)\| \\ &\geqslant \inf_{f \in F} \| e - f \| = d(e, F) = 1,\end{aligned}$$

so $d(x) = \| x \| = 1$. Choose any $t_0 \in T$ and choose a continuous function $f: T \to [0, 1]$ so that $f(t_0) = 1$ and f vanishes off a compact set. Set $y = (-f)\, x$. Then $y \in M$ (indeed, $y(\infty) = 0$), $\sigma y = y$, $\sigma x = z(x) = 0$, $d(x + y) = d(x) = 1$, and $z(x + y) = z(x) = 0$ imply

$$\begin{aligned}\sigma(x + y)(t_0) &= z(x + y) = 0 \neq -e \\ &= y(t_0) = \sigma x(t_0) + \sigma y(t_0). \quad \blacksquare\end{aligned}$$

3.5. *Remark.* It is not possible in general to choose a *linear* selection for P_M. For if it were, then by specializing so that $C(T, E) = \ell_\infty$ and $C_F(T, E) = c_0$ (i.e. take $T = \mathbb{N}$, $E = \mathbb{R}$, and $F = \{0\}$), it would follow that this selection would be a continuous linear projection from ℓ_∞ onto c_0, hence implying c_0 is complemented in ℓ_∞, which is not the case.

Also, part (7) shows that σ is not even "additive modulo M". This is in sharp contrast to the metric projection itself which always has this property:

$$P_M(x + y) = P_M x + P_M y$$

for each $x \in X$, $y \in M$.

We now show that the selection σ satisfies a certain extremal property.

3.6. PROPOSITION. *For every $x \in X$ and $t \in T$,*

$$\| \sigma x(t) - z(x)\| = \min\{\| y(t) - z(x)\| : y \in Px\}.$$

In particular,

$$\| \sigma x - z(x)\| = \min\{\| y - z(x)\| : y \in Px\}.$$

Proof. If $\sigma x(t) = z(x)$ there is nothing to prove. If not, then $\|\sigma x(t) - z(x)\| = \|x(t) - z(x)\| - d(x)$, while for every $y \in Px$ we have

$$\|x(t) - z(x)\| - \|y(t) - z(x)\| \leqslant \|x(t) - y(t)\| \leqslant d(x)$$

so that

$$\|\sigma x(t) - z(x)\| = \|x(t) - z(x)\| - d(x) \leqslant \|y(t) - z(x)\|. \quad ∎$$

3.7. PROPOSITION. *If $x \in X \setminus M$, then for every $y \in M$ there exists $\lambda \geqslant 0$ and y_1, y_2 in Px such that $y = \lambda(y_1 - y_2 - y(\infty))$. Thus $M = \text{cone}(Px - Px) + F$.*

Proof. Let $x' \in Px$. Then $Px - Px = P(x - x' - z(x)) - P(x - x' + z(x))$ and $z(x) \in Px - x' + z(x) = P(x - x' + z(x))$. Therefore we may assume $z(x) \in Px$, i.e. $\|x - z(x)\| = d(x)$. By scaling y, we may assume $\|y\| \leqslant \frac{1}{2}d(x)$ and hence that $\|y - y(\infty)\| \leqslant d(x)$. Define

$$\begin{aligned} y_1(t) &= z(x) + y(t) - y(\infty) \qquad \text{if } \|x(t) - z(x) - y(t) - y(\infty)\| \leqslant d(x) \\ &= x(t) - \frac{d(x)}{\|x(t) - z(x) - y(t) + y(\infty)\|}[x(t) - z(x) - y(t) + y(\infty)] \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{otherwise.} \end{aligned}$$

Clearly $y_1 \in X$. Given $\epsilon > 0$, choose $K \in \mathscr{K}$ with $\|y(t) - y(\infty)\| < \epsilon/2$ off K. Let $t \notin K$. If $\|x(t) - z(x) - y(t) + y(\infty)\| \leqslant d(x)$, then $\|y_1(t) - z(x)\| = \|y(t) - y(\infty)\| < \epsilon/2 < \epsilon$. If $\|x(t) - z(x) - y(t) + y(\infty)\| > d(x)$, then $\|y_1(t) - z(x) - y(t) + y(\infty)\| = \|x(t) - z(x) - y(t) + y(\infty)\| - d(x) \leqslant \|x(t) - z(x)\| + \|y(t) - y(\infty)\| - d(x) \leqslant \|y(t) - y(\infty)\| < \epsilon/2$. Thus $\|y_1(t) - z(x)\| \leqslant \|y_1(t) - z(x) - y(t) + y(\infty)\| + \|y(t) - y(\infty)\| < \epsilon$. This shows that $\lim_{t \to \infty} y_1(t) = z(x)$, hence $y_1 \in M$. Also clearly $\|x(t) - y_1(t)\| \leqslant d(x)$ for all t so that $\|x - y_1\| \leqslant d(x)$ and $y_1 \in Px$.

Set $y_2 = y_1 - y + y(\infty)$. Then $y_2 \in M$. If $\|x(t) - z(x) - y(t) + y(\infty)\| \leqslant d(x)$, then $\|x(t) - y_2(t)\| = \|x(t) - z(x)\| \leqslant d(x)$. If $\|x(t) - z(x) - y(t) + y(\infty)\| > d(x)$, then

$$\begin{aligned} \|x(t) - y_2(t)\| &\leqslant \frac{d(x)}{\|x(t) - z(x) - y(t) + y(\infty)\|}\|x(t) - z(x)\| \\ &\qquad + \left[1 - \frac{d(x)}{\|x(t) - z(x) - y(t) + y(\infty)\|}\right]\|y(t) - y(\infty)\| \\ &\leqslant \frac{d(x)^2}{\|x(t) - z(x) - y(t) + y(\infty)\|} \\ &\qquad + \left[1 - \frac{d(x)}{\|x(t) - z(x) - y(t) + y(\infty)\|}\right] d(x) = d(x). \end{aligned}$$

Thus $y_2 \in Px$.

3.8. *Remark.* We cannot, in general, discard $y(\infty)$. E.g. when $x(t) = e$, $e \in E \backslash F$, then $z(x) = P_F e$ and $d(x) = d(e, F)$, so that for every $y \in Px$ we must have $y(\infty) = P_F e = z(x)$. Thus cone$(Px - Px)$ cannot be all of M unless $F = \{0\}$.

3.9. PROPOSITION. *If $x, y \in X$ and $y' \in Py$, then $x' \equiv y' + \sigma(x - y') \in Px$ and $\| x' - y' \| \leqslant 2 \| x - y \| + 2 \sup_{y'' \in Py} \| z(x - y'')\|$. In particular,*

$$d_H(Px, Py) \leqslant 2 \| x - y \| + 2 \max\{ \sup_{y'' \in Py} \| z(x - y'')\|, \sup_{x'' \in Px} \| z(y - x'')\|\}.$$

Proof. $x - x' = x - y' - \sigma(x - y')$ so that $\| x - x' \| = d(x - y') = d(x)$ and $x' \in Px$. Also

$$\begin{aligned} \| x' - y' \| &= \| \sigma(x - y')\| \leqslant \| \sigma(x - y') - z(x - y')\| + \| z(x - y')\| \\ &\leqslant \| x - y' - z(x - y')\| - d(x - y') + \| z(x - y')\| \end{aligned}$$

(using 3.4(5))

$$\begin{aligned} &\leqslant \| x - y' \| + 2 \| z(x - y')\| - d(x) \\ &\leqslant \| x - y \| + \| y' - y \| + 2 \| z(x - y')\| - d(x) \\ &= \| x - y \| + 2 \| z(x - y')\| + d(y) - d(x) \\ &\leqslant 2 \| x - y \| + 2 \| z(x - y')\| \\ &\leqslant 2 \| x - y \| + 2 \sup_{y'' \in Py} \| z(x - y'')\|. \end{aligned}$$

By symmetry, for each $x' \in Px$ there is $y' \in Py$ so that

$$\| x' - y' \| \leqslant 2 \| x - y \| + 2 \sup_{x'' \in Px} \| z(y - x'')\|.$$

The last statement in the proposition follows easily from these two inequalities. ∎

3.10. *Remark.* The second term in the upper bound for $d_H(Px, Py)$ cannot be dropped in general. For let E be uniformly convex and F be a closed subspace. If we take $x(t) = u$ and $y(t) = v$ for all t, then for each $x' \in Px$, $y' \in Py$ we easily deduce that $x'(\infty) = P_F u$, $y'(\infty) = P_F v$, and $\| x' - y' \| \geqslant \| x'(\infty) - y'(\infty)\| = \| P_F u - P_F v \|$. Thus $d_H(Px, Py) \geqslant \| P_F u - P_F v \|$. If the second term in the upper bound for $d_H(Px, Py)$ could be dropped, it would follow that P_F is Lipschitz continuous. But this is false in general (cf. Holmes and Kripke [6], example 5).

4. The $C_0(T, E)$ Case and M-Ideals

In this section we specialize the results of Section 3 to the case when E is any normed space and $F = \{0\}$. That is, we consider approximation in $C(T, E)$ by $C_0(T, E)$, the subspace of continuous E-valued functions on T vanishing at infinity : $x \in C_0(T, E)$ iff $x \in C(T, E)$ and for each $\epsilon > 0$, $\{t \in T : \|x(t)\| \geqslant \epsilon\}$ is compact. (We leave to the reader the simple exercise of specializing the results of this section even futher to obtain the important case of approximation in ℓ_∞ by the subspace c_0 .)

4.1. Proposition. *Let T be any locally compact noncompact space, E any normed linear space, $X = C(T, E)$, $M = C_0(T, E)$, and $P = P_M$. Then:*

(1) *For every $x \in X$, $d(x) = d(x, M) = \lim_{t\to\infty} \sup \|x(t)\|$ $(= \inf_{K\in\mathscr{K}} \sup_{t\in T\setminus K} \|x(t)\|)$;*

(2) *For every $x \in X\setminus M$, $M = \text{cone}(Px - Px)$. In fact, for each $y \in M$ with $\|y\| \leqslant \frac{1}{2}d(x)$, $y = x' - x''$ for some $x', x'' \in Px$;*

(3) *$d_H(Px, Py) \leqslant 2\|x - y\|$ for each $x, y \in X$ and 2 is the best constant;*

(4) *P_M is Hausdorff continuous and lower semicontinuous but not upper semicontinuous at any point of $X\setminus M$;*

(5) *$P^{-1}(0) = \{x \in X : 0 \in Px\}$ is nowhere dense.*

Proof. (1) is a consequence of Proposition 3.1 since

$$\inf_{K\in\mathscr{K}} r_0(x(T\setminus K)) = \inf_{K\in\mathscr{K}} \sup_{t\in T\setminus K} \|x(t)\|.$$

(2) is a particular case of Proposition 3.7 and its proof.

(3) is a particular case of Proposition 3.9 In order to see that the constant 2 is best possible, take any $e \in E$ with $\|e\| = 1$, fix $t_0 \in T$ and a compact neighborhood K of t_0 , and choose $f \in C(T, \mathbb{R})$ to satisfy $f(T\setminus K) = 0 \leqslant f \leqslant 1 = f(t_0)$. Set $x(t) = [f(t) + \frac{1}{2}]\, e$, $y(t) = 2f(t)\, e$. Then $x \in X$, $y \in M$, $\|x - y\| = \frac{1}{2}$, $d(x) = \frac{1}{2}$, $x'(t) = f(t)\, e$ is in Px, but

$$\begin{aligned} d_H(Px, Py) &= d_H(Px, y) \geqslant \|x' - y\| \geqslant \|x'(t_0) - y(t_0)\| \\ &= \|f(t_0)\, e\| = 1 = 2\|x - y\|. \end{aligned}$$

(4) The Hausdorff continuity follows from (3), while the lower semicontinuity follows from lower Hausdorff-semicontinuity which, in turn, follows from Hausdorff continuity (cf. [8], [4]). To show that P is not upper semicontinuous at any point $x \in X\setminus M$, it suffices (by [4], Theorem 1) to show that Px is not compact. But if Px were compact, so would be $Px - Px$

which, by (2) contains the ball in M of radius $\frac{1}{2}d(x)$. Thus M must be finite dimensional which is not the case.

(5) follows from Proposition 3.2. ∎

Observe that the same argument which proved 4.1(4), combined with a previously mentioned result of Holmes, Scranton, and Ward [7], shows that 4.1(4) *is valid for every M-ideal M in an arbitrary normed linear space X.*

We also obtain

4.2. COROLLARY. *The elements in $X = C(T, E)$ which attain their norm form a dense set in X.*

Proof. By 4.1(5), it suffices to show that if $x \in C(T, E)$ does not attain its norm, then $x \in P_M^{-1}(0)$. Fix any $t_0 \in T$. Since $\| x(t_0)\| < \| x \|$, for each compact set $K \in \mathscr{K}$, there exists $t \notin K$ such that $\| x(t)\| > \| x(t_0)\|$. Hence $\sup_{t \notin K} \| x(t)\| > \| x(t_0)\|$ implies that

$$d(x) = \inf_{K \in \mathscr{K}} \sup_{t \notin K} \| x(t)\| \geqslant \| x(t_0)\|.$$

Since $t_0 \in T$ was arbitrary, $d(x) \geqslant \| x \|$ and hence $x \in P_M^{-1}(0)$. ∎

Holmes, Scranton, and Ward [7] had proved the analogue of Corollary 4.2 when $X = \mathscr{B}(H)$, the bounded linear operators on a Hilbert space H.

4.3. PROPOSITION. *Let T, X, M, P, and d(x) be as in Proposition* 4.1. *Then the function σ defined on X by*:

$$\begin{aligned}(\sigma x)(t) &= 0 && \text{if} \quad \| x(t)\| \leqslant d(x)\\ &= \left[1 - \frac{d(x)}{\| x(t)\|}\right] x(t) && \text{otherwise,}\end{aligned}$$

has the following properties:

(1) *σ is a homogeneous selection for the metric projection P*;

(2) *σ satisfies the Lipschitz condition* $\| \sigma x - \sigma y \| \leqslant 2 \| x - y \|$, *and* 2 *is the best constant*;

(3) $\| \sigma x \| \leqslant \| x \| - d(x)$, *and* $\| \sigma x \| = \| x \|$ *iff* $x \in M$;

(4) *σ is minimal in norm, i.e.* $\| \sigma x \| = \min\{\| y \| : y \in Px\}$ *for each* $x \in X$. *This even holds pointwise, i.e.* $\| \sigma x(t)\| = \min\{\| y(t)\| : y \in Px\}$ *for each* $x \in X$, $t \in T$;

(5) *σ is not additive*; *in fact, σ is not even additive modulo M, i.e. there exist* $x \in X$ *and* $y \in M$ *with* $\sigma(x + y) \neq \sigma x + \sigma y$.

Proof. The only addition to the results of Section 3 in this particular case is the second statement in (2). But this follows from the same example we used in Proposition 4.1(3): the x' there is just σx. ∎

Some of the results of Proposition 4.1, as well as the *existence* of a selection for P having certain desirable properties, follow from the general theory of "*M-ideals*". Recall that a closed subspace M of a Banach space X is called an *M-ideal* if there is a (linear) projection Q of X^* onto the annihilator $M^\perp$ of M in X^* such that $\| x^* \| = \| Qx^* \| + \| x^* - Qx^* \|$ for all $x^* \in X^*$, i.e $M^\perp$ is an L-summand in X^*. (For the definitions, properties, and characterization of M-ideals, see Alfsen and Effros [1].)

Fakhoury [5] gave a *non*constructive proof, using Michael's selection theorem, of the existence of a continuous homogenous selection for the metric projection P_M onto an M-ideal M in X. Holmes, Scranton, and Ward [7] proved that if M is an M-ideal in X, then $d_H(P_M x, P_M y) \leqslant 2 \| x - y \|$ for all x, y in X and span $P_M x = M$ for every $x \in X \backslash M$. They asked "for which M-ideals M is it true that $P_M^{-1}(0)$ has no interior?" They showed this to be the case for $c_0 \subset \ell_\infty$ and the compact operators $\mathscr{C}(H) \subset \mathscr{B}(H)$ on a Hilbert space H (and false for "M-summands"). A partial answer to their to their question is given by 4.1(5) and the following proposition.

4.4. Proposition. *$C_0(T, E)$ is an M-ideal in $C(T, E)$.*

Proof. If we want to avoid representation theorems for $C(T, E)^*$, we may use the following characterization of M-ideals by the "3-ball property" [1] : For each x_1, x_2, x_3, x in X, ϵ, $r_1, r_2, r_3 > 0$, and y_1, y_2, y_3 in M with $\| x_i - y_i \| < r_i - \epsilon$, $\| x_i - x \| < r_i - \epsilon$ $(i = 1, 2, 3)$, there is a $y \in M$ with $\| x_i - y \| < r_i$ $(i = 1, 2, 3)$.

Indeed, take a compact set $K \subset T$ such that $\| y_i(t) \| < \epsilon$ off K, and then $\| x_i(t) \| < r_i$ off K. Take a compact neighborhood K_1 of K and $f \in C(T)$ with $f(T \backslash K_1) = 0 \leqslant f \leqslant 1 = f(K)$. Let $y(t) = f(t)\, x(t)$. Then $y \in M$ and $\| x_i - y \| < r_i$ $(i = 1, 2, 3)$. ∎

We note that the results in 4.1 and 4.3 are stronger than those obtained by using M-ideal theory, and their proofs are elementary. Moreover, the results of Section 3 do *not* follow from the M-ideal theory, as can be seen by comparing *Remark* 3.10 above and Theorem 2 of [7], or directly from

4.5. Proposition. (1) *If the Stone-Čech compactification of T, βT, is not the one-point compactification T^*, then $C_F(T, E)$ is an M-ideal in $C(T, E)$ if and only if $F = \{0\}$;*

(2) *If $\beta T = T^*$ and E is finite-dimensional, then $C_F(T, E)$ is an M-ideal in $C(T, E)$ if and only if F is an M-ideal in E. In the general case, if $C_F(T, E)$ is an M-ideal in $C(T, E)$, then F is an M-ideal in E.*

Proof. (1) The "if" part was proved in Proposition 4.4. For the other half, take $z_0 \in F$ with $\| z_0 \| = 1$, and $q_1 \neq q_2$ in $\beta T \backslash T^*$. Choose any continuous $\tilde{g} : \beta T \backslash T^* \to [-1, 1]$ with $\tilde{g}(q_1) = -1 \leqslant \tilde{g} \leqslant 1 = \tilde{g}(q_2)$. By Tietze's theorem we can extend $\tilde{g}$ to a continuous function $g : T \to [-1, 1]$. Let $x_1(t) = g(t)\, z_0$, $x_2(t) = [g(t) + 1]\, z_0$, $x_3(t) = [g(t) - 1]\, z_0$, $y_1(t) = 0$, $y_2(t) = r_0$, and $y_3(t) = -z_0$. Then for every $0 < \epsilon < 1$, $\| x_i - y_i \| < 1 + \epsilon$ and $\| x_i - x_1 \| < 1 + \epsilon$ $(i = 1, 2, 3)$. But if $y \in F$ satisfies $\| x_i - y \| < 1 + \epsilon$ $(i = 2, 3)$, we must have $\| 2z_0 - y(\infty)\| < 1 + \epsilon$ and $\| 2z_0 + y(\infty)\| < 1 + \epsilon$; hence $4 = \| 4z_0 \| < 2 + 2\epsilon < 4$, a contradiction.

(2) In this case $C(T, E) = C(\beta T, E)$. The annihilator of $C_F(T, E)$ is $F^\perp \nu_\infty$, i.e. the $F^\perp$-valued point measures at ∞, and this is an L-summand in $C(T, E)^* =$ the E^*-valued measures on T^* if and only if $F^\perp$ is an L-summand in E^*, i.e. iff F is an M-ideal in E.

If F is not an M-ideal in E, take e_1, e_2, e_3 in E, g_1, g_2, g_3 in F, and $r_1, r_2, r_3, \epsilon > 0$ which fails the 3-ball property and consider the constant functions $x_i(t) = e_i$, $y_i(t) = g_i$ in $C(T, E)$ and $C_F(T, E)$, respectively, which fails the 3-ball property. ∎

Holmes, Scranton, and Ward [7] asked "When does the following equation hold:

$$\bigcap \{P_M x : x \in P_M^{-1}(0), \| x \| = 1\} = \{0\}\text{?"} \tag{*}$$

They showed this to be case for the M-ideal of compact operators $\mathscr{C}(H)$ in $\mathscr{B}(H)$ and the M-ideal c_0 in ℓ_∞. (They also claimed that (*) fails for "M-summands.")

We answer their question by proving that (*) *always* holds.

4.6. Proposition. *If X is any normed linear space and M any proximinal subspace, then* (*) *holds.*

Proof. Take any $y \in M \backslash \{0\}$ and any $x \in P_M^{-1}(0)$ with $\| x \| = 1$. If $y \in P_M x$, let $z = \alpha y$, where $\alpha = \sup\{\beta \geqslant 0 : \beta y \in P_M x\} \geqslant 1$. Then $z \in P_M x$ (since $P_M x$ is closed), hence $0 \in P_M(x - z)$, $d(x - z, M) = d(x, M) = 1$, and thus $\| x - z \| = 1$. If $y \in P_M(x - z)$, then

$$y \in P_M x - z = P_M x - \alpha y \text{ implies } (1 + \alpha)\, y \in P_M x$$

which contradicts the choice of α. Thus $y \notin P_M(x - z)$. ∎

In the proof we actually showed that if $y \in M \backslash \{0\}$, then there exist x_1, x_2 in $P_M^{-1}(0)$, $\| x_1 \| = \| x_2 \| = 1$, such that $y \notin P_M x_1 \cap P_M x_2$. In some cases, we may even take $x_2 = -x_1$. Indeed, taking $X = (\mathbb{R} \times \mathbb{R})_\infty$, $M = \mathbb{R} \times \{0\}$, and $x = (1, 1)$, we get $P_M x \cap P_M(-x) = \{0\}$.

A natural question then is : For which proximinal subspaces M of a normed space X is it true that there exists an $x \in P_M^{-1}(0)$, $\|x\| = 1$, such that $P_M x \cap P_M(-x) = \{0\}$?

The answer is clearly affirmative if M is a Chebyshev subspace or even if *some* point in $X \backslash M$ has a unique best approximation in M. Indeed, the following result is a complete characterization.

4.7. PROPOSITION. *Let M be a proximinal subspace of a normed space X. The following are equivalent for an element $x \in P_M^{-1}(0)$ with $\|x\| = 1$:*

(1) $P_M x \cap P_M(-x) = \{0\}$;

(2) *x is a relatively M-extreme point of the unit ball in X, i.e. x is not the midpoint of a line segment in the unit ball which is parallel to M.*

Proof. (1) $\Rightarrow$ (2). If (2) fails, we can write $x = \frac{1}{2}(y_1 + y_2)$, where $\|y_i\| \leqslant 1$ and $y = y_1 - y_2 \in M \backslash \{0\}$. By replacing y_i, y by $y_i' = \frac{1}{2}(y_i + x)$ and $y' = y_1' - y_2'$ if necessary, we may assume that $\|x \pm y\| \leqslant 1$ and hence $y \in P_M x \cap P_M(-x)$. Thus (1) fails.

(2) $\Rightarrow$ (1). Let $y \in P_M(x) \cap P_M(-x)$. Then $\|x \pm y\| = d(x, M) = 1$. Setting $y_1 = x + y$, $y_2 = x - y$, we obtain $\|y_i\| = 1$, $y_1 - y_2 \in M$: and $x = \frac{1}{2}(y_1 + y_2)$. Since x is relatively M-extreme, $y_1 = y_2$, i.e. $y = 0$. ∎

In particular, the question also has an affirmative answer whenever the set ext $B(x) \cap P_M^{-1}(0)$ is nonempty. (Here ext $B(X)$ denotes the set of extreme points of the unit ball in X.) As a corollary of this remark, we obtain

4.8. COROLLARY. *If the unit ball in E has an extreme point, $X = C(T, E)$, and $M = C_0(T, E)$, then there is an $x \in P_M^{-1}(0)$ with $\|x\| = 1$ and $P_M x \cap P_M(-x) = \{0\}$.*

Proof. Let $e \in \text{ext } B(E)$ and $x(t) = e$ for all t. Since $\|x\| = 1$ and $d(x, M) = 1$, $x \in P_M^{-1}(0)$. Thus it suffices to show $x \in \text{ext } B(X)$. If not, there exists $y \in X \backslash \{0\}$ such that $\|x \pm y\| \leqslant 1$. Then $\|y\| \leqslant 1$ and $\|e \pm y(t)\| \leqslant 1$ for all t. Since e is extreme, $y(t) = 0$ for all t, a contradiction. ∎

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